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On the evaluation of statical-kinematic stiffness matrix for underconstrained structural systems

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Abstract

Because of singularity of the elastic stiffness matrix, analysis of underconstrained systems requires the comprehensive stiffness matrix, comprising both kinds of first-order structural stiffness—elastic and statical-kinematic. The latter accounts for the role of the member forces, induced in the system by the equilibrium part of the applied load, as the source of the system resistance to the perturbation part of the same load. Evaluation of the sought member forces using orthogonal load resolution into equilibrium and perturbation components is awkward and computationally expensive. A special modification of the singular elastic stiffness matrix simplifies the procedure and makes it amenable to the conventional tools of structural analysis. The efficiency and accuracy of the procedure is illustrated by a detailed numerical example.

1. Introduction

This note is a companion to a recent paper by the author (Kuznetsov, 1997) where the comprehensive stiffness matrix for an underconstrained structural system has been introduced as the sum of the conventional elastic stiffness matrix and the statical-kinematic stiffness matrix:

$$C_{mn} = E_{mn} + K_{mn} = F^i_m S_{ij} F^j_n + F^i_{mn} \Lambda_i \tag{1}$$

The notation (employing the summation convention for a repeated index) is as follows.

$$F^i_n = \partial F^i / \partial X_{n|0}, \quad F^i_{mn} = \partial^2 F^i / \partial X_m \partial X_{n|0} \tag{2}$$

are, respectively, the $C \times N$ constraint function Jacobian matrix and the set of C constraint function Hessian matrices obtained from the set of constraint equations of the system,

$$F^i(X_1, \ldots, X_n, \ldots, X_N; L_i) = 0, \quad i = 1, 2, \ldots, C. \tag{3}$$

The C constraint functions F^i relate the N generalized coordinates, X_n, to the known geometric

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parameters, L_i (linear and angular sizes of the structural members). At least one solution to the constraint equations, $X_n = X_n^0$, must exist and is assumed known; it is taken as the reference geometric configuration. Since the equilibrium matrix for a given configuration is the Jacobian matrix transpose, the equilibrium equations for the reference configuration are

$$F_n^i \Lambda_i = P_n^*. \tag{4}$$

Here P_n^* is an external load with N components acting in the directions of the respective coordinates, X_n. To be statically possible, the load must be in the column space of the equilibrium matrix, in which case it is called an equilibrium load. The generalized constraint reactions, Λ_i, depend on the form of constraint functions (3); for a pin-bar assembly with the constraint equations expressing inextensibility conditions for the bars, Λ_i are simply the bar forces.

In terms of constraint eqns (3), elongations of structural members are interpreted as the constraint variations, f_i, related to the system displacements, x_n, by the kinematic relations

$$F_n^i x_n = f_i. \tag{5}$$

If elongations are elastic, the member forces are expressed with the aid of the diagonal matrix of member stiffnesses, S_{ij}:

$$\Lambda_i = S_{ij} f_j = S_{ij} F_n^j x_n. \tag{6}$$

Substituting this into eqn (4) leads to the system of equilibrium equations in displacements

$$F_m^i S_{ij} F_n^j x_n \equiv E_{mn} x_n = P_m^* \tag{7}$$

where the elastic stiffness matrix, E_{mn}, has the same rank, r, as the matrix F_n^i; thus, for an underconstrained structural system ($r<N$), the elastic stiffness matrix is singular. Although round-off errors in coordinate digitization and other numerical operations are likely to produce instead a nonsingular matrix, it is very ill-conditioned; for the purposes of this discussion, such matrices will be referred to, and treated as, singular.

Singularity of the elastic stiffness matrix is a sign of inadequacy of the linear elastic model. Physically, it means that the system cannot support certain loads without large displacements. Algebraically, this indicates the existence of the Jacobian matrix nullspace, i.e., the space of inextensible displacements, x_n^0, representing a nontrivial, indefinite magnitude, solution to the homogeneous system of kinematic equations

$$F_n^i x_n = 0 \tag{8}$$

A conventional way of restraining inextensible displacements is to account for stiffness imparted to the system by pre-existing stress state. Indeed, any displacement from a state of equilibrium alters the nodal resultants of the pre-existing member forces, thereby giving rise to 'product forces' (Calladine, 1982; Pellegrino and Calladine, 1986). These forces are usually accounted for in any incremental geometrically nonlinear analysis, even one with a nonsingular elastic stiffness matrix.

The statical-kinematic stiffness matrix K_{mn} in eqn (1) refines the conventional nonlinear analysis and makes it possible in the absence of initial forces produced by prestress or pre-existing loads. This matrix is based on the orthogonal load resolution whereby a general load P_n is resolved into its equilibrium and perturbation components: P_n^* in the column space of the equilibrium matrix, and p_n in its orthogonal complement space (the left nullspace), so that

$$P_n^* + p_n = P_n, \quad P_n^* p_n = 0. \tag{9}$$

Such a resolution always exists, is unique and local (configuration-specific).

An equilibrium load is balanced by a structural system in a given configuration without any displacements (assuming stability and perfectly rigid members). On the other hand, under a perturbation load, equilibrium in the original configuration is impossible and displacements are unavoidable regardless of the elastic properties of the system. The reason is that perturbation loads and inextensible displacements are in the nullspace of the singular elastic stiffness matrix, which rules out a unique solution for displacements.

Some software packages contain remedies for dealing with an ill-conditioned elastic stiffness matrix, like the single point constraint option or the soft spring option. Introducing such fictitious constraints, rigid or soft, expands the column space of the matrix, restores its rank, and makes solution possible. The introduced constraints develop reactions equilibrating the perturbation part of the applied load, i.e., the part that the system cannot resist in its original configuration. Rigid fictitious constraints produce the necessary reactions without any deflection, thus preserving the system configuration (except for the elastic deformations); flexible constraints may have to undergo very large deflections before they equilibrate the perturbation part of the load. In either case (rigid or flexible constraints), the imparted resistance to the perturbation load is fictitious and the obtained solution may be grossly inaccurate. For underconstrained systems, such a solution is often unsuitable for either an iterative refinement or as the first step of an incremental procedure.

The basic idea underlying the matrix K_{mn} is to account for the role of the equilibrium part of the applied load as the source of the system resistance to the perturbation part of the same load. In fact, this represents the actual physical mechanism enabling an underconstrained structural system to support a general load. For a given equilibrium load, the system resistance to perturbation loads is expressed by the positive definite (assuming stable equilibrium) tangent stiffness matrix K_{mn}. The matrix is purely statical-kinematic in nature: it is determined solely by the current geometry of the system and the member forces induced by the equilibrium part of the applied load. Note that the elastic properties of the system are irrelevant in the context of perturbation loads.

The definitive feature of the statical-kinematic stiffness matrix is the forward dependence on the equilibrium part of the external load (as opposed to a pre-existing load or prestress). However, the matrix use requires knowing the member forces induced by the equilibrium part of the load.

2. Modified elastic stiffness matrix

A direct approach to obtaining the member forces produced by the equilibrium part of the applied load involves a computationally expensive procedure of pseudoinverse (Strang, 1988). Worse still, for underconstrained systems with simultaneous statical and kinematic indeterminacies (Tarnai, 1980), the analysis requires orthogonal load resolution followed by solving a statically indeterminate problem for the equilibrium part of the applied load—an awkward procedure alien to the conventional finite element methodology and tools (Kuznetsov, 1997).

It turns out that instead of using pseudoinverse or the orthogonal load resolution, the sought member forces can be evaluated by solving a system of equations wherein the elastic stiffness matrix is modified in a certain way. The resulting procedure utilizes only conventional tools of the finite element analysis and statical indeterminacy does not affect it.

The singular elastic stiffness matrix E_{mn} is modified by adding to its diagonal elements a very small quantity, ε, of the same unit dimension:

$$E^*_{mn} = E_{mn} + \varepsilon I_{mn} \tag{10}$$

where I_{mn} is the identity matrix. Clearly, the stiffness matrix so modified is no longer singular.

The displacements of the modified system under a given load P_m are obtained from

$$E^*_{mn} x_n = P_m. \tag{11}$$

These displacements can be presented as a combination of inextensible and elastic displacements,

$$x_n = x^0_n + x^*_n, \tag{12}$$

which are, respectively, in the nullspace of the original singular elastic stiffness matrix and in its orthogonal complement space (Volokh and Vilnay, 1997). Inextensible displacements, x^0_n, produce rigid body translations and rotations of structural members (no elongations), whereas elastic displacements, x^*_n, produce member elongations and the resulting internal forces. Accordingly, the member elongations are obtainable from the kinematic relations (5) (the linearized constraint equations) as the constraint variations, f_i, corresponding to displacements (12); the inextensible displacements are filtered out in the process by virtue of eqn (8),

$$f_i = F^i_n x_n = F^i_n x^*_n, \tag{13}$$

so that the resulting member forces are still given by eqn (6).

An external load equilibrated by the modified system with displacements (12), can be separated into two components, attributable, respectively, to the inextensible and elastic displacements:

$$E^*_{mn} x^0_n = E_{mn} x^0_n + \varepsilon x^0_m = p_m, \tag{14}$$

$$E^*_{mn} x^*_n = E_{mn} x^*_n + \varepsilon x^*_m = P^*_m. \tag{15}$$

The first product in the right hand side of eqn (14) vanishes because of eqns (7) and (8). The second term admits a serendipitous interpretation rooted in the statical-kinematic duality: as a load following the pattern of inextensible displacements, it represents a perturbation load for the original system (inextensible displacements and perturbation loads span one and the same space—the null space of the constraint Jacobian matrix). Both terms in the right of eqn (15) are multiples of the elastic displacement vector x^*_n which is in the column space of the original equilibrium matrix, hence, is an equilibrium load for the original system.

According to eqn (11), wherefrom the above displacements have been obtained, the sum of the perturbation and equilibrium loads, given respectively by eqns (14) and (15), equals the given load P_m. Thus, the modified stiffness matrix in eqn (10) implicitly performs the orthogonal load resolution, making unnecessary an evaluation of the equilibrium part of the applied load and solution of equilibrium equations for this load.

In physical terms, amending the singular stiffness matrix in eqn (10) amounts to imposing very flexible, identical elastic constraints on all N nodal degrees of freedom of the system. Since the perturbation part of the applied load cannot be supported by the system, it is resisted solely by the introduced constraints, at the expense of their (perhaps, large) displacements. Remarkably, with all of the imposed constraints having the same stiffness ε, their reaction pattern described by eqn (14) represents exactly the perturbation part, p_n, of the applied load. This follows from the fact that p_n mimics the pattern of the inextensible displacements (which makes it a perturbation load) and the fact that the rest of the load is an equilibrium load.

The equilibrium load in eqn (15) is balanced by the nodal resultants of the member forces produced in the modified system by displacements x_n^*. Since these are elastic displacements both for the given system and for the array of imposed flexible constraints, load P_m^* is shared by the two subsystems in proportion to their respective stiffnesses, as seen from the right hand side of eqn (15). If the chosen magnitude of ε is sufficiently small, the equilibrium part of the applied load is supported almost entirely by the given system, with the contribution of the imposed array of flexible constraints confined to a designated error tolerance. Thus, the flexible constraints enable the modified system to support a general load without materially affecting the member forces corresponding to the equilibrium part of the applied load. According to eqns (6) and (8), the sought member forces are found using displacements x_n obtained by solving eqn (11):

$$\Lambda_i = S_{ij} F_n^j x_n^* = S_{ij} F_n^j x_n. \tag{16}$$

With these forces available, the statical-kinematic stiffness matrix can be assembled and added to the elastic stiffness matrix to form the comprehensive stiffness matrix, eqn (1).

Although the outlined approach involves only linear operations, it amounts to a two-stage semi-nonlinear analysis, due to the described forward dependence feature. The approach can be used as a typical step in either an iterative or incremental nonlinear analysis. Note one particular situation that defies the proposed procedure by making a nonlinear problem nonlinearizable. This happens when the applied load does not contain an equilibrium part, i.e., is a pure perturbation load. Then the statical-kinematic stiffness matrix vanishes, leaving the comprehensive tangent stiffness matrix singular. A possible practical remedy is to add a fictitious equilibrium load (e.g., by introducing an arbitrary set of member forces and evaluating their nodal resultants) and then to remove this load in one or several incremental steps.

3. Example, observations and discussion

Consider a plane underconstrained pin-bar system consisting of two symmetric polygons connected with two vertical posts (Fig. 1). Its geometric configuration is defined by $N = 8$ coordinates of the four nodes and although the number of constraints (bars) is also $C = 8$, the system is not geometrically invariant but only quasi-invariant. Accordingly, the equilibrium matrix is singular not due to the lack of

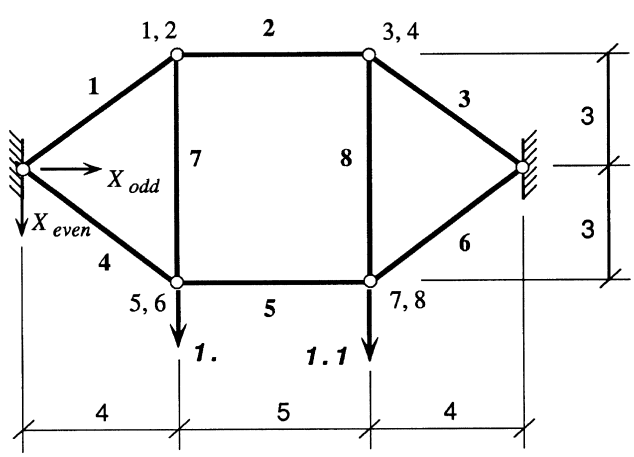

Fig. 1. Underconstrained quasi-invariant pin-bar system.

Table 1
Singular elastic stiffness matrix for the example system

1640	−480	−1000	0	0	0	0	0
−480	1193	0	0	0	−833	0	0
−1000	0	1640	480	0	0	0	0
0	0	480	1193	0	0	0	−833
0	0	0	0	1640	480	−1000	0
0	−833	0	0	480	1193	0	0
0	0	0	0	−1000	0	1640	−480
0	0	0	−833	0	0	−480	1193

Table 2
Applied load and its mutually orthogonal equilibrium and perturbation parts

P_n	0	0	0	0	0	1	0	1.1
P_n^*	0.012	0.016	0.012	−0.016	−0.012	1.016	−0.012	1.084
p_n	−0.012	−0.016	−0.012	0.016	0.012	−0.016	0.012	0.016

constraints, but because of the singular geometry. As a result, the system admits a single state of self-stress with tension in the upper and lower chords and compression in the posts. The self-stress, being a nontrivial solution to the homogeneous equilibrium equations, indicates that the equilibrium matrix and, with it, the elastic stiffness matrix, are singular. The singular configuration is physically realizable by prestressing which is assumed of some finite but negligibly small magnitude.

After specifying the system geometry with suitable integer bar lengths shown in Fig. 1, the singular, 8×8 equilibrium matrix is formed; the matrix rank is $r = 7$ so that degree of statical indeterminacy of the system is $S = C - r = 1$. The elastic stiffness matrix (Table 1) is evaluated assuming axial stiffnesses of all bars identical, $EA = 5000$; this matrix is also singular, with the rank $r = 7$.

Consider a general load shown in Fig. 1. The horizontal and vertical nodal components of the load, along with the components of its equilibrium and perturbation parts (obtained by orthogonal resolution), are given in Table 2.

Note that the explicit orthogonal load resolution has been carried out solely for the purpose of the intended comparisons; the main objective of this paper is to present and explore computational

Table 3
Three alternative solutions for displacements and forces

Single point constraint		Load resolution		Modified stiffness matrix	
Displacements	Forces	Displacements	Forces	Displacements	Forces
−0.00100	−0.7994	−0.00092	−0.7369108	−12000	−0.7369109
0	−0.6396	4.66E−10	−0.6015287	−16000	−0.6015288
−0.00164	−0.7994	−0.00152	−0.7669108	−12000	−0.7669107
0.00352	1.0339	0.00330	0.9830892	16000	0.9830891
0.00095	0.8271	0.00085	0.7984713	12000	0.7984713
0.00046	1.0339	0.00052	1.0130892	−16000	1.0130891
0.00178	0.3797	0.00164	0.4261465	12000	0.4261463
0.00409	0.4797	0.00388	0.4761465	16000	0.4761462

alternatives to load resolution. Still, there are some advantages to having an explicit expression for the perturbation load. First, since the number of linearly independent perturbation loads for any system is $V = N - r$, for the example system there exists only one such load. Thus, in this case the perturbation part of any external load P_n is determined by just one scale factor. Second, since this single perturbation load mimics the displacement pattern of the inextensible deformation mode, the latter is obtained as a by-product of evaluating the former (and vice-versa).

For the given general load (Fig. 1), an exact linear solution to the problem does not exist. Three approximate solutions have been obtained and compared, judging by the effort required and results produced. The first solution utilizes a single point constraint and the resulting system of equations with nonsingular stiffness matrix. The second solution is obtained by separating and applying the equilibrium part of the given load; since this load is in the column space of the original singular stiffness matrix, the obtained equations are compatible and solvable by a suitable numerical method. The result is the best possible (least square) approximate solution for the original, incompatible, system of equations with the given load in the right hand side. Finally, the modified stiffness matrix approach based on eqn (11) has been implemented.

The described three solutions are presented for comparison in Table 3. In the first solution, the vertical displacement at node 1 (dof no. 2) has been restrained by a rigid support. In the second solution (the singular stiffness matrix and equilibrium load), a very soft spring for dof no. 2 was introduced to control the inextensible displacements which are, strictly speaking, of zero magnitude. The third solution employs eqn (11) with the modified stiffness matrix. In all three cases the member forces are evaluated from the obtained displacements by using eqn (6).

In the first solution, the reaction of the introduced rigid constraint is exactly $R = -0.1$. Although the effective load (inclusive of R) applied to the system is an equilibrium load, it is not the equilibrium part of the given load (cf Table 2). In the second solution, with the equilibrium load obtained by the orthogonal resolution, the reaction of the soft spring, $R = 5\text{E} - 16$, is strictly a numerical noise. This illustrates a statical property of independent constraints (Kuznetsov, 1991). For a system under an equilibrium load, imposition of an independent constraint has no effect on the force distribution and the introduced constraint is force-free. (Note that introduction of an independent constraint into a geometrically invariant system is impossible). Finally, in the third solution, the singular stiffness matrix was modified according to eqn (10), with $\varepsilon = 1\text{E} - 6$.

Comparing the presented solutions and results leads to the following observations:

(1) The computational expenses of the first and third solutions are practically the same whereas the second one, involving the orthogonal load resolutions, is the most expensive.
(2) The first solution yields reasonably looking, but not very accurate displacements and even worse member forces. This is the result of subjecting the system to a wrong effective equilibrium load (a combination of the given load with the reaction in the introduced rigid support).

Table 4
Composition of statical-kinematic stiffness matrix

$d_1+d_2+d_7$	0	$-d_2$	0	$-d_7$	0	0	0
0	$d_1+d_2+d_7$	0	$-d_2$	0	$-d_7$	0	0
$-d_2$	0	$d_2+d_3+d_8$	0	0	0	$-d_8$	0
0	$-d_2$	0	$d_2+d_3+d_8$	0	0	0	$-d_8$
$-d_7$	0	0	0	$d_4+d_5+d_7$	0	$-d_5$	0
0	$-d_7$	0	0	0	$d_4+d_5+d_7$	0	$-d_5$
0	0	$-d_8$	0	$-d_5$	0	$d_5+d_6+d_8$	0
0	0	0	$-d_8$	0	$-d_5$	0	$d_5+d_6+d_8$

Table 5
Comprehensive stiffness matrix

1639.8	−480	−999.9	0	−0.071	0	0	0
−480	1192.8	0	0.1203	0	−833.1	0	0
−999.9	0	1639.8	480	0	0	−0.079	0
0	0.1203	480	1192.8	0	0	0	−833.1
−0.071	0	0	0	1640.4	480	−1000	0
0	−833.1	0	0	480	1193.4	0	−0.16
0	0	−0.079	0	−1000	0	1640.4	−480
0	0	0	−833.1	0	−0.16	−480	1193.4

Table 6
Displacements, member forces, and updated loads

Displacements	−0.1153	−0.1524	−0.1159	0.1557	0.1151	−0.1519	0.1159	0.1562
Forces	−0.7518	−0.6009	−0.7518	0.9473	0.7967	1.0484	0.5106	0.3901
Loads	−0.0029	0.0004	0.0030	0.0005	0.0024	1.0012	−0.0025	1.1010
Load P_n	0	0	0	0	0	1.0	0	1.1

(3) The displacements in the third solution are so predominantly inextensible that their elastic components (of the same order as in the other two solutions) are unobservable within the number of decimal places displayed.
(4) Most importantly, the member forces in the third solution coincide to within six decimal places with those of the best (least square) solution based on the orthogonal load resolution. (This is consistent with the above value of the parameter ε.) Thus, a practically identical evaluation of the member forces sought is achieved in the third solution at a lower computational expense.

With the member forces produced by the equilibrium part of the given load available, the statical-kinematic stiffness matrix in eqn (1) can be constructed as a combination of constraint function Hessians weighted by the constraint reactions. Using the notation $d_i = \Lambda_i/L_i$, this symmetric matrix, given in Table 4, reflects the system connectivity.

The statical-kinematic stiffness matrix, evaluated using the member forces from the third solution in Table 3, is combined with the singular elastic stiffness matrix (Table 1) to form the comprehensive stiffness matrix, eqn (1), presented in Table 5.

Upon solving the obtained system of equations for displacements, the member forces are evaluated from eqn (6). The outcome is presented in the first two rows of Table 6.

The member forces are fairly close to their previously calculated values in the last two solutions of Table 3; this is an illustration of the accuracy of the least square solution. Although the system displacements under the given load are predominantly inextensible, they are very different from, and are much more accurate than, the displacements in any of the three solutions in Table 3. Furthermore, the displacement magnitudes, compared to the system dimensions, provide a good clue to the overall accuracy of the analysis. For the present example, the achieved accuracy has been assessed by updating the system geometry (using the displacements from Table 6) and evaluating the corresponding new nodal resultants of the same member forces. The found nodal force resultants, presented in the third row of Table 6, ideally must balance the applied load P_n (the last row in the Table). Their comparison suggests that the accuracy of the presented analysis is reasonable; if judged otherwise, an iterative or an incremental nonlinear analysis is in order.

4. Concluding remarks

Comprehensive stiffness matrix, combining the two sources of first-order structural stiffness—elastic and statical-kinematic—is the most general tangent stiffness matrix for a linear (or linearized) analysis. Conventional incremental procedures account only for the previously existing and already known member forces produced by prestress and/or prior loading. This approach is usually adequate in a nonlinear analysis of geometrically invariant systems, where the tangent elastic stiffness matrix is nonsingular, the elastic stiffness is predominant, and even a relatively large error in quantifying the statical-kinematic stiffness can be tolerated.

The situation with underconstrained structural systems is diametrically opposite. Here the statical-kinematic stiffness is the only source of the system resistance to perturbation loads; it remedies the singularity of the elastic stiffness matrix by providing a solution within its null space (the space of inextensible displacements). Hence, an accurate evaluation of the statical-kinematic stiffness is necessary, accounting, in particular, for the member forces induced by the equilibrium part of the currently applied load (or load increment). This forward dependence on the yet unknown member forces correctly reflects mechanics of deformation and is the principal feature of the statical-kinematic stiffness matrix. In contrast, the conventional techniques, lacking this feature, are not only less accurate but, in certain cases, may prove inadequate; for example, they require some adaptations when dealing with an underconstrained system without initial forces.

The forward dependence of statical-kinematic stiffness on the member forces produced by the equilibrium part of the applied load requires knowing these forces in advance. Their calculation, using the pseudoinverse for statically determinate systems and the orthogonal load resolution for indeterminate ones, is computationally expensive and awkward. The presented procedure, employing the modified elastic stiffness matrix, streamlines the calculation and makes it amenable to the common computing tools of structural analysis, thereby facilitating the use of the existing software in the analysis of underconstrained systems.

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